# $p$-ADIC GAMMA FUNCTION AND TRACES OF FROBENIUS OF ELLIPTIC CURVES

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**Abstract**: In [10], McCarthy defined a function ${}_nG_n[\cdots]$ using Teichmüller character of finite fields and quotients of $p$-adic gamma function, and expressed the trace of Frobenius of elliptic curves in terms of special values of ${}_2G_2[\cdots]$. We establish two different expressions for the trace of Frobenius of elliptic curves in terms of the function ${}_2G_2[\cdots]$. As a result, we obtain two relations between special values of the function ${}_2G_2[\cdots]$ with different parameters.

**Key Words**: Trace of Frobenius; elliptic curves; characters of finite fields; Gauss sums; Teichmüller character; $p$-adic Gamma function.

2010 Mathematics Classification Numbers: Primary: 11G20, 33E50; Secondary: 33C99, 11S80, 11T24.

## 1. Introduction and statement of results

Let $q = p^r$ be a power of an odd prime and let $\mathbb{F}_q$ be the finite field of $q$ elements. Let $\Gamma_p(.)$ denote the Morita's $p$-adic gamma function and let $\omega$ denote the Teichmüller character of $\mathbb{F}_q$. We denote by $\overline{\omega}$ the inverse of $\omega$. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$ and $\langle x \rangle$ denote the fractional part of $x$, i.e. $x - \lfloor x \rfloor$. Also, we denote by $\mathbb{Z}^+$ and $\mathbb{Z}_{\geq 0}$ the set of positive integers and non negative integers, respectively. In [10], McCarthy defined a function ${}_nG_n[\cdots]$ as given below.

**Definition 1.1.** [10, Defn. 5.1] Let $q = p^r$, for $p$ an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i$, $b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function ${}_nG_n[\cdots]$ is defined by

$$
{}_nG_n\left[\begin{array}{cccc} a_1, & a_2, & \cdots, & a_n \\ b_1, & b_2, & \cdots, & b_n \end{array} | t\right]_q := \frac{-1}{q-1}\sum_{j=0}^{q-2}(-1)^{jn}\,\overline{\omega}^j(t)
$$
$$
\times \prod_{i=1}^{n}\prod_{k=0}^{r-1}(-p)^{-\lfloor\langle a_i p^k\rangle - \frac{jp^k}{q-1}\rfloor - \lfloor\langle -b_i p^k\rangle + \frac{jp^k}{q-1}\rfloor}\frac{\Gamma_p(\langle(a_i - \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle a_i p^k\rangle)}\frac{\Gamma_p(\langle(-b_i + \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle -b_i p^k\rangle)}.
$$

Throughout this paper we will refer to this function as ${}_nG_n[\cdots]$. This function has many interesting properties. In [4], Greene introduced the notion of hypergeometric functions over finite fields. Since then, many interesting connections between hypergeometric functions over finite field and algebraic curves have been found. But these results are restricted to primes satisfying certain congruence conditions. For example, see [1, 2, 3, 8, 9]. Let $E/\mathbb{F}_q$ be an elliptic curve given in the Weierstrass form. Then the trace of Frobenius $a_q(E)$ of $E$ is given by

$$
a_q(E) := q + 1 - \#E(\mathbb{F}_q), \tag{1}
$$

where $\#E(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-points on $E$ including the point at infinity. Let $j(E)$ denote the $j$-invariant of the elliptic curve $E$. Let $\phi$ be the quadratic character of $\mathbb{F}_q^\times$ extended to all of $\mathbb{F}_q$ by setting $\phi(0) := 0$. Using the function ${}_2G_2[\cdots]$, McCarthy expressed the trace of Frobenius of elliptic curves without any congruence condition on the prime. The statement of his result is given below.

**Theorem 1.2.** [10, Thm. 1.2] *Let $p > 3$ be a prime. Consider an elliptic curve $E_s/\mathbb{F}_p$ of the form $E_s : y^2 = x^3 + ax + b$ with $j(E_s) \neq 0, 1728$. Then*

$$a_p(E_s) = \phi(b) \cdot p \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \mid -\frac{27b^2}{4a^3}\right]_p. \tag{2}$$

In [2], the first author & Kalita gave two formulas for the trace of Frobenius of the elliptic curve $E_{a,b} : y^2 = x^3 + ax + b$ defined over $\mathbb{F}_q$ under the conditions $q \equiv 1 \pmod 6$ and $q \equiv 1 \pmod 4$, respectively. In this paper, we prove the following two expressions for the trace of Frobenius of the elliptic curve $E_{a,b}/\mathbb{F}_q$ in terms of special values of the function ${}_2G_2[\cdots]$ without any congruence conditions on $q$.

**Theorem 1.3.** *Let $q = p^r$, $p > 3$ be a prime. Consider an elliptic curve $E_{a,b}/\mathbb{F}_q$ of the form $E_{a,b} : y^2 = x^3 + ax + b$ with $j(E_{a,b}) \neq 0$. If $(-a/3)$ is a quadratic residue in $\mathbb{F}_q$, then*

$$a_q(E_{a,b}) = \phi(k^3 + ak + b) \cdot q \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \mid -\frac{k^3 + ak + b}{4k^3}\right]_q,$$

*where $3k^2 + a = 0$.*

**Theorem 1.4.** *Let $q = p^r$, $p > 3$ be a prime. Consider an elliptic curve $E_{a,b}/\mathbb{F}_q$ of the form $E_{a,b} : y^2 = x^3 + ax + b$ with $j(E_{a,b}) \neq 1728$. If $x^3 + ax + b = 0$ has a non zero solution in $\mathbb{F}_q$, then*

$$a_q(E_{a,b}) = \phi(-3h^2 - a) \cdot q \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4} \end{matrix} \mid \frac{4(3h^2 + a)}{9h^2}\right]_q,$$

*where $h^3 + ah + b = 0$.*

McCarthy proved the Theorem 1.2 over $\mathbb{F}_p$ and remarked that the result could be generalized for $\mathbb{F}_q$. Along the proof of the Theorem 1.3 and Theorem 1.4 (which are proved for $\mathbb{F}_q$), we have verified that the Theorem 1.2 is in fact true for $\mathbb{F}_q$. Hence, we have the following corollary which gives nice transformation formulas between special values of the function ${}_2G_2[\cdots]$ with different parameters.

**Corollary 1.5.** *Let $q = p^r$, $p > 3$ be a prime. Let $a, b \in \mathbb{F}_q^\times$. Then*

$${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \mid -\tfrac{27b^2}{4a^3}\right]_q = \begin{cases} \phi(b(k^3 + ak + b)) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3} \end{matrix} \mid -\frac{k^3+ak+b}{4k^3}\right]_q & \text{if } a = -3k^2; \\ \phi(-b(3h^2 + a)) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4} \end{matrix} \mid \frac{4(3h^2+a)}{9h^2}\right]_q & \text{if } h^3 + ah + b = 0. \end{cases}$$

## 2. Preliminaries

In this section, we recall some results which we will use to prove our main results. We start defining the additive character $\theta : \mathbb{F}_q \to \mathbb{C}^\times$ by

$$\theta(\alpha) = \zeta^{\mathrm{tr}(\alpha)}$$

where $\zeta = e^{2\pi i/p}$ and $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map given by

$$\mathrm{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters of $\mathbb{F}_q^\times$. We extend the domain of each $\chi \in \mathbb{F}_q^\times$ to $\mathbb{F}_q$ by setting $\chi(0) := 0$ including the trivial character $\varepsilon$. For $A \in \widehat{\mathbb{F}_q^\times}$, the *Gauss sum* is defined by

$$G(A) := \sum_{x\in\mathbb{F}_q} A(x)\zeta^{\mathrm{tr}(x)} = \sum_{x\in\mathbb{F}_q} A(x)\theta(x).$$

We let $T$ denote a fixed generator of $\widehat{\mathbb{F}_q^\times}$. The Gauss sum $G(T^m)$ is denoted by $G_m$. The following lemma provides a formula for the multiplicative inverse of a Gauss sum.

**Lemma 2.1.** ([4, Eqn. 1.12]). *If $k \in \mathbb{Z}$ and $T^k \neq \varepsilon$, then*

$$G_k G_{-k} = qT^k(-1).$$

The *orthogonality relations* for multiplicative characters are listed in the following lemma.

**Lemma 2.2.** ([6, Chapter 8]). *We have*

(1) $$\sum_{x\in\mathbb{F}_q} T^n(x) = \begin{cases} q-1 & \text{if } T^n = \varepsilon; \\ 0 & \text{if } T^n \neq \varepsilon. \end{cases}$$

(2) $$\sum_{n=0}^{q-2} T^n(x) = \begin{cases} q-1 & \text{if } x = 1; \\ 0 & \text{if } x \neq 1. \end{cases}$$

Using orthogonality, we can write $\theta$ in terms of Gauss sums as given in the following lemma.

**Lemma 2.3.** ([3, Lemma 2.2]). *For all $\alpha \in \mathbb{F}_q^\times$,*

$$\theta(\alpha) = \frac{1}{q-1}\sum_{m=0}^{q-2} G_{-m}T^m(\alpha).$$

**Theorem 2.4.** (Davenport-Hasse Relation [7]). *Let $m$ be a positive integer and let $q = p^r$ be a prime power such that $q \equiv 1 (mod\ m)$. For multiplicative characters $\chi, \psi \in \widehat{\mathbb{F}_q^\times}$, we have*

$$\prod_{\chi^m=1} G(\chi\psi) = -G(\psi^m)\psi(m^{-m}) \prod_{\chi^m=1} G(\chi). \tag{3}$$

Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$. Let $\mathbb{Z}_q$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let $\omega : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$ be the Teichmüller character. For $a \in \mathbb{F}_q^\times$, the value $\omega(a)$ is just the $(q-1)$-th root of unity in $\mathbb{Z}_q$ such that $\omega(a) \equiv a \pmod{p}$. We denote by $\overline{\omega}$ the inverse of $\omega$. We now recall the $p$-adic gamma function. For $n \in \mathbb{Z}^+$, the $p$-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n, p\nmid j} j$$

and one extends it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n)$$

for $x \neq 0$, where $x$ runs through any sequence of positive integers $p$-adically approaching $x$. This limit exists, is independent of how $n$ approaches $x$, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^{\times}$. We now state a product formula for the $p$-adic gamma function. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x$ satisfies $0 \leq x \leq 1$ and $(q-1)x \in \mathbb{Z}$, then

$$\prod_{h=0}^{m-1} \Gamma_p\left(\frac{x+h}{m}\right) = \omega\left(m^{(1-x)(1-p)}\right) \Gamma_p(x) \prod_{h=1}^{m-1} \Gamma_p\left(\frac{h}{m}\right). \tag{4}$$

We also note that

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0}, \tag{5}$$

where $x_0 \in \{1, 2, \ldots, p\}$ satisfies $x_0 \equiv x \pmod p$. The Gross-Koblitz formula allow us to relate the Gauss sums and the $p$-adic gamma function. Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. Then we have the following result.

**Theorem 2.5.** (Gross, Koblitz [5]). *For $a \in \mathbb{Z}$ and $q = p^r$,*

$$G(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1}\langle \frac{ap^i}{q-1}\rangle} \prod_{i=0}^{r-1} \Gamma_p\left(\langle \frac{ap^i}{q-1} \rangle\right).$$

## 3. Proof of the results

We first prove a lemma which we will use to prove the main results.

**Lemma 3.1.** *Let $p$ be a prime and $q = p^r$. For $0 \leq j \leq q-2$, $0 \leq i \leq r-1$ and $t \in \mathbb{Z}^+$ with $p \nmid t$, we have*

$$\Gamma_p\left(\langle \frac{tp^ij}{q-1} \rangle\right) \omega\left(t^{\frac{tp^ij}{\sum_{s=0}^{r-1} p^s}}\right) \prod_{h=1}^{t-1} \Gamma_p\left(\langle \frac{hp^i}{t} \rangle\right) = \prod_{h=0}^{t-1} \Gamma_p\left(\langle \frac{p^ih}{t} + \frac{p^ij}{q-1} \rangle\right) \tag{6}$$

*and*

$$\Gamma_p\left(\langle \frac{-tp^ij}{q-1} \rangle\right) \omega\left(t^{\frac{-tp^ij}{\sum_{s=0}^{r-1} p^s}}\right) \prod_{h=1}^{t-1} \Gamma_p\left(\langle \frac{hp^i}{t} \rangle\right) = \prod_{h=0}^{t-1} \Gamma_p\left(\langle \frac{p^i(1+h)}{t} - \frac{p^ij}{q-1} \rangle\right). \tag{7}$$

*Proof.* Fix $0 \leq j \leq q-2$, $0 \leq i \leq r-1$ and let $k \in \mathbb{Z}_{\geq 0}$ be defined such that

$$k\left(\frac{q-1}{t}\right) \leq jp^i < (k+1)\left(\frac{q-1}{t}\right) \text{ and } \frac{k}{p^i} \in \mathbb{Z}_{\geq 0}. \tag{8}$$

Putting $m = t$ and $x = \frac{tp^ij}{q-1} - k$ in (4), we obtain

$$\prod_{h=0}^{t-1} \Gamma_p\left(\frac{p^ij}{q-1} + \frac{h-k}{t}\right) = \omega\left(t^{\,(1-\frac{tp^ij}{q-1}+k)(1-p)}\right) \Gamma_p\left(\frac{tp^ij}{q-1} - k\right) \prod_{h=1}^{t-1} \Gamma_p\left(\frac{h}{t}\right). \tag{9}$$

We observe that $0 \leq \frac{k}{p^i} < t$. Using (8) we see that if $0 \leq h < t$, then $0 \leq \frac{h-k}{t} + \frac{p^i j}{q-1} < 1$. Therefore, if $1 \leq \frac{k}{p^i} < t$ then

$$\prod_{h=0}^{t-1} \Gamma_p \left( \frac{h-k}{t} + \frac{p^i j}{q-1} \right) = \prod_{h=0}^{t-1} \Gamma_p \left( \left\langle \frac{h-k}{t} + \frac{p^i j}{q-1} \right\rangle \right) = \prod_{h=0}^{t-1} \Gamma_p \left( \left\langle \left( \frac{h-k}{p^i t} + \frac{j}{q-1} \right) p^i \right\rangle \right). \tag{10}$$

Letting $\frac{h}{p^i} = z$ and $\frac{k}{p^i} = l$ in (10) we get

$$\begin{aligned} &\prod_{h=0}^{t-1} \Gamma_p \left( \frac{h-k}{t} + \frac{p^i j}{q-1} \right) \\ &= \prod_{z=0}^{l-1} \Gamma_p \left( \left\langle \left( \frac{z-l}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \prod_{z=l}^{t-1} \Gamma_p \left( \left\langle \left( \frac{z-l}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \\ &= \prod_{z=0}^{l-1} \Gamma_p \left( \left\langle \left( \frac{t+z-l}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \prod_{z=l}^{t-1} \Gamma_p \left( \left\langle \left( \frac{z-l}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \\ &= \prod_{z=t-l}^{t-1} \Gamma_p \left( \left\langle \left( \frac{z}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \prod_{z=0}^{t-l-1} \Gamma_p \left( \left\langle \left( \frac{z}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \\ &= \prod_{z=0}^{t-1} \Gamma_p \left( \left\langle \left( \frac{z}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right) \\ &= \prod_{h=0}^{t-1} \Gamma_p \left( \left\langle \left( \frac{h}{t} + \frac{j}{q-1} \right) p^i \right\rangle \right). \end{aligned} \tag{11}$$

The expression (11) also holds for $k = 0$. Again by our choice of $k$ we have

$$\Gamma_p \left( \left\langle \frac{p^i t j}{q-1} \right\rangle \right) = \Gamma_p \left( \frac{t p^i j}{q-1} - k \right). \tag{12}$$

Also,

$$\begin{aligned} \omega \left( t^{\,(1 - \frac{t p^i j}{q-1} + k)(1-p)} \right) &= \omega \left( t^{(1+k)(1-p)} t^{\,\frac{t p^i j (p-1)}{(q-1)}} \right) \\ &= \omega \left( t^{\,\frac{t p^i j}{1+p+p^2+\cdots+p^{r-1}}} \right). \end{aligned} \tag{13}$$

Now substituting (11), (12), (13) into (9) we obtain (6).

We prove (7) following [10, Lemma 4.1] and using similar arguments as given in the proof of (6). $\square$

**Lemma 3.2.** *For $1 \leq l \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\begin{aligned}&\lfloor -\frac{lp^i}{q-1}\rfloor - \lfloor -\frac{2lp^i}{q-1}\rfloor - \lfloor -\frac{2lp^i}{q-1}\rfloor - \lfloor \frac{3lp^i}{q-1}\rfloor - 1\\ &= -2\lfloor \langle \frac{p^i}{2}\rangle - \frac{lp^i}{q-1}\rfloor - \lfloor \langle -\frac{p^i}{3}\rangle + \frac{lp^i}{q-1}\rfloor - \lfloor \langle -\frac{2p^i}{3}\rangle + \frac{lp^i}{q-1}\rfloor.\end{aligned}$$

*Proof.* We can prove the lemma by considering the following cases:

Case 1: $1 \leq l < \frac{q-1}{6p^i}$.
Case 2: $\lfloor \frac{q-1}{6p^i}\rfloor < l \leq q-2$.
In case 2, we observe that $\lfloor \frac{6lp^i}{q-1}\rfloor = 1, 2, \ldots, (6p^i-1)$. Now taking $x \in \mathbb{Z}^+$ such that $1 \leq x \leq 6p^i - 1$ and $x = 6u + v$, where $v = 0, 1, 2, 3, 4$ or $5$, the result follows. □

**Lemma 3.3.** *For $0 \leq l \leq q-2$ and $0 \leq i \leq r-1$, we have*

$$\begin{aligned}&\lfloor \frac{2lp^i}{q-1}\rfloor + 2\lfloor \frac{-lp^i}{q-1}\rfloor - 2\lfloor \frac{-2lp^i}{q-1}\rfloor - \lfloor \frac{4lp^i}{q-1}\rfloor\\ &= -2\lfloor \langle \frac{p^i}{2}\rangle - \frac{lp^i}{q-1}\rfloor - \lfloor \langle -\frac{p^i}{4}\rangle + \frac{lp^i}{q-1}\rfloor - \lfloor \langle -\frac{3p^i}{4}\rangle + \frac{lp^i}{q-1}\rfloor.\end{aligned}$$

*Proof.* We can prove the lemma by considering the following cases:

Case 1: $0 \leq l < \frac{q-1}{4p^i}$.
Case 2: $\lfloor \frac{q-1}{4p^i}\rfloor < l \leq q-2$.
In case 2, we observe that $\lfloor \frac{4lp^i}{q-1}\rfloor = 1, 2, \ldots, (4p^i-1)$. Now taking $x \in \mathbb{Z}^+$ such that $1 \leq x \leq 4p^i - 1$ and $x = 4u + v$, where $v = 0, 1, 2$ or $3$, the desired result follows. □

Now, we are going to prove Theorem 1.3. The proof will follow as a consequence of the next theorem. We consider an elliptic curve $E_1$ over $\mathbb{F}_q$ in the form

$$E_1 : y^2 = x^3 + cx^2 + d,$$

where $c \neq 0$. We express the trace of Frobenius endomorphism on the curve $E_1$ as a special value of the function ${}_2G_2[\cdots]$ in the following way.

**Theorem 3.4.** *Let $q = p^r$, $p > 3$ be a prime. The trace of Frobenius on $E_1$ is given by*

$$a_q(E_1) = q \cdot \phi(d) \cdot {}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{3}, & \frac{2}{3}\end{matrix} \mid -\frac{27d}{4c^3}\right]_q.$$

*Proof.* We have $\#E_1(\mathbb{F}_q) - 1 = \#\{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q : y^2 = x^3 + cx^2 + d\}$.
Let $P(x,y) = x^3 + cx^2 + d - y^2$. Now using the identity

$$\sum_{z \in \mathbb{F}_q} \theta(zP(x,y)) = \begin{cases} q, & \text{if } P(x,y) = 0;\\ 0, & \text{if } P(x,y) \neq 0,\end{cases} \tag{14}$$

we obtain

$$\begin{aligned}
q.(\#E_1(\mathbb{F}_q)-1) &= \sum_{x,y,z\in\mathbb{F}_q}\theta(zP(x,y))\\
&= q^2+\sum_{z\in\mathbb{F}_q^\times}\theta(zd)+\sum_{y,z\in\mathbb{F}_q^\times}\theta(zd)\theta(-zy^2)+\sum_{x,z\in\mathbb{F}_q^\times}\theta(zd)\theta(zx^3)\theta(zcx^2)\\
&\quad+\sum_{x,y,z\in\mathbb{F}_q^\times}\theta(zd)\theta(zx^3)\theta(zcx^2)\theta(-zy^2)\\
&= q^2+A+B+C+D. \qquad (15)
\end{aligned}$$

From the proof of [2, Theorem 3.1], we have $A=-1$, $B=1+qT^{\frac{q-1}{2}}(d)$ and $D=-C+D_{\frac{q-1}{2}}$ , where

$$\begin{aligned}
D_{\frac{q-1}{2}} = \frac{1}{(q-1)^3}&\sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}G_{\frac{q-1}{2}}T^l(d)T^n(c)T^{\frac{q-1}{2}}(-1)\\
&\sum_{x\in\mathbb{F}_q^\times}T^{3m+2n}(x)\sum_{z\in\mathbb{F}_q^\times}T^{l+m+n+\frac{q-1}{2}}(z),
\end{aligned}$$

which is non zero only if $m=-\frac{2}{3}n$ and $n=-3l-\frac{3(q-1)}{2}$. Since $G_{3l+\frac{3(q-1)}{2}}=G_{3l+\frac{q-1}{2}}$ and $G_{-2l-(q-1)}=G_{-2l}$, we have

$$D_{\frac{q-1}{2}} = \frac{1}{q-1}\sum_{l=0}^{q-2}G_{-l}G_{-2l}G_{3l+\frac{q-1}{2}}G_{\frac{q-1}{2}}T^l(d)T^{-3l+\frac{q-1}{2}}(c)T^{\frac{q-1}{2}}(-1). \qquad (16)$$

Replacing $l$ by $l-\frac{q-1}{2}$ we obtain

$$\begin{aligned}
D_{\frac{q-1}{2}} &= \frac{1}{q-1}\sum_{l=0}^{q-2}G_{-l+\frac{q-1}{2}}G_{-2l}G_{3l}G_{\frac{q-1}{2}}T^{l-\frac{q-1}{2}}(d)T^{-3l}(c)T^{\frac{q-1}{2}}(-1)\\
&= \frac{\phi(-d)}{q-1}\sum_{l=0}^{q-2}G_{-l+\frac{q-1}{2}}G_{-2l}G_{3l}G_{\frac{q-1}{2}}T^l(d)T^{-3l}(c). \qquad (17)
\end{aligned}$$

Using Davenport-Hasse relation (Theorem 2.4) for $m=2$, $\psi=T^{-l}$, we deduce that

$$G_{-l+\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}G_{-2l}T^l(4)}{G_{-l}}. \qquad (18)$$

Substituting (18) into (17) and using lemma 2.1 we deduce that

$$D_{\frac{q-1}{2}} = \frac{q\phi(d)}{q-1}\sum_{l=0}^{q-2}\frac{G_{-2l}G_{-2l}G_{3l}}{G_{-l}}T^l\left(\frac{4d}{c^3}\right).$$

Putting the values of $A, B, C$ and $D$ in (15) we obtain

$$q\cdot(\#E_1(\mathbb{F}_q)-1) = q^2+q\phi(d)+D_{\frac{q-1}{2}},$$

which yields

$$a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}\frac{G_{-2l}G_{-2l}G_{3l}}{G_{-l}}T^l\left(\frac{4d}{c^3}\right). \tag{19}$$

Now we take $T$ to be the inverse of the Teichmüller character, i.e., $T=\overline{\omega}$ and use the Gross-Koblitz formula (Theorem 2.5) to convert the above expression to an expressing involving the $p$-adic gamma function. This gives

$$a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\{\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{3lp^i}{q-1}\rangle-\langle\frac{-lp^i}{q-1}\rangle\}}$$
$$\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{3lp^i}{q-1}\rangle\right)}{\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)}\overline{\omega}^l\left(\frac{4d}{c^3}\right).$$

If we put $s=\displaystyle\sum_{i=0}^{r-1}\left\{\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{3lp^i}{q-1}\rangle-\langle\frac{-lp^i}{q-1}\rangle\right\}$, then the above equation becomes

$$a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^s\overline{\omega}^l\left(\frac{4d}{c^3}\right)$$
$$\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{3lp^i}{q-1}\rangle\right)}{\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)}. \tag{20}$$

Next we use lemma 3.1 and simplify (20) to obtain

$$a_q(E_1) = -\phi(d) - \frac{\phi(d)}{q-1}\sum_{l=0}^{q-2}(-p)^s\overline{\omega}^l\left(\frac{27d}{4c^3}\right)$$
$$\times\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(1-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{l}{q-1}\right)p^i\right\rangle\right)$$
$$\times\frac{\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)}$$
$$\times\frac{\Gamma_p\left(\left\langle\left(\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{3}\rangle\right)\Gamma_p\left(\langle\frac{2p^i}{3}\rangle\right)}. \tag{21}$$

Calculating $s$ we deduce that

$$s=\sum_{i=0}^{r-1}\left\{\lfloor\frac{-lp^i}{q-1}\rfloor-\lfloor\frac{-2lp^i}{q-1}\rfloor-\lfloor\frac{-2lp^i}{q-1}\rfloor-\lfloor\frac{3lp^i}{q-1}\rfloor\right\}. \tag{22}$$

By (5) we have that, for $0 \leq l \leq q-2$,

$$\prod_{i=0}^{r-1} \Gamma_p\left(\left\langle\left(1-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{l}{q-1}\right)p^i\right\rangle\right) = (-1)^r\overline{\omega}^l(-1). \tag{23}$$

Therefore,

$$\begin{aligned} a_q(E_1) &= -\frac{q\phi(d)}{q-1} - \frac{q\phi(d)}{q-1}\sum_{l=1}^{q-2}(-p)^{s-r}\overline{\omega}^l\left(-\frac{27d}{4c^3}\right) \\ &\times \prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)} \\ &\times \frac{\Gamma_p\left(\left\langle\left(\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{3}\rangle\right)\Gamma_p\left(\langle\frac{2p^i}{3}\rangle\right)}. \end{aligned}$$

Now using the following relation for $0 \leq l \leq q-2$

$$\begin{aligned} &\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right) \\ &= \prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(-\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right) \end{aligned} \tag{24}$$

and lemma 3.2, we deduce that

$$\begin{aligned} a_q(E_1) &= -\frac{q\phi(d)}{q-1}\sum_{l=0}^{q-2}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{p^i}{3}\rangle+\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{2p^i}{3}\rangle+\frac{lp^i}{q-1}\rfloor} \\ &\times \frac{\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)} \\ &\times \frac{\Gamma_p\left(\left\langle\left(-\frac{1}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{2}{3}+\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle-\frac{p^i}{3}\rangle\right)\Gamma_p\left(\langle-\frac{2p^i}{3}\rangle\right)}\overline{\omega}^l\left(-\frac{27d}{4c^3}\right) \\ &= q\cdot\phi(d)\cdot {}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ \frac{1}{3}, & \frac{2}{3}\end{matrix}\ \middle|\ -\frac{27d}{4c^3}\right]_q. \end{aligned}$$

This completes the proof of the theorem. □

**Proof of Theorem 1.3:** We have $j(E_{a,b}) \neq 0$. Hence $a \neq 0$. Since $(-a/3)$ is a quadratic residue in $\mathbb{F}_q$, we find $k \in \mathbb{F}_q^\times$ such that $3k^2 + a = 0$. A change of variables $(x, y) \mapsto (x + k, h)$ takes the elliptic curve $E_{a,b} : y^2 = x^3 + ax + b$ to

$$E' : y^2 = x^3 + 3kx^2 + (k^3 + ak + b). \tag{25}$$

Clearly $a_q(E_{a,b}) = a_q(E')$. Using Theorem 3.4 for the elliptic curve $E'$, we complete the proof.

Now, we are going to prove Theorem 1.4. The proof will follow as a consequence of the next theorem. We consider an elliptic curve $E_2$ over $\mathbb{F}_q$ in the form

$$E_2 : y^2 = x^3 + fx^2 + gx,$$

where $f \neq 0$. We express the trace of Frobenius endomorphism on the curve $E_2$ as a special value of the function ${}_2G_2[\cdots]$ in the following way.

**Theorem 3.5.** *Let $q = p^r$, $p > 3$ be a prime. The trace of Frobenius on $E_2$ is given by*

$$a_q(E_2) = q \cdot \phi(-g) \cdot {}_2G_2\left[\begin{array}{cc} \frac{1}{2}, & \frac{1}{2} \\ \frac{1}{4}, & \frac{3}{4} \end{array} \mid \frac{4g}{f^2}\right]_q.$$

*Proof.* We recall that $\#E_2(\mathbb{F}_q) - 1 = \#\{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q : y^2 = x^3 + fx^2 + gx\}$.
Let $P(x, y) = x^3 + fx^2 + gx - y^2$. Now using the identity

$$\sum_{z\in\mathbb{F}_q} \theta(zP(x, y)) = \begin{cases} q, & \text{if } P(x, y) = 0; \\ 0, & \text{if } P(x, y) \neq 0, \end{cases} \tag{26}$$

we obtain

$$\begin{aligned} q \cdot (\#E_2(\mathbb{F}_q) - 1) &= \sum_{x,y,z\in\mathbb{F}_q} \theta(zP(x, y)) \\ &= q^2 + \sum_{z\in\mathbb{F}_q^\times} \theta(0) + \sum_{y,z\in\mathbb{F}_q^\times} \theta(-zy^2) + \sum_{x,z\in\mathbb{F}_q^\times} \theta(zx^3)\theta(zfx^2)\theta(zgx) \\ &+ \sum_{x,y,z\in\mathbb{F}_q^\times} \theta(zx^3)\theta(zfx^2)\theta(zgx)\theta(-zy^2) \\ &= q^2 + (q - 1) + A + B + C. \end{aligned} \tag{27}$$

From [2, Theorem 3.2] we have $A = -(q - 1)$ and $C = -B + C_{\frac{q-1}{2}}$, where

$$\begin{aligned} C_{\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}}{(q-1)^3} &\sum_{l,m,n=0}^{q-2} G_{-l}G_{-m}G_{-n}T^m(f)T^n(g)T^{\frac{q-1}{2}}(-1) \\ \times &\sum_{x\in\mathbb{F}_q^\times} T^{3l+2m+n}(x) \sum_{z\in\mathbb{F}_q^\times} T^{l+m+n+\frac{q-1}{2}}(z), \end{aligned}$$

which is non zero only if $n = l$ and $m = -2l + \frac{q-1}{2}$. This gives

$$C_{\frac{q-1}{2}} = \frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1} \sum_{l=0}^{q-2} G_{-l}G_{2l+\frac{q-1}{2}}G_{-l}T^l\left(\frac{g}{f^2}\right).$$

Substituting the values of $A$, $B$ and $C$ in (27) we obtain

$$q \cdot (\#E_2(\mathbb{F}_q) - 1) = q^2 + \frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1} \sum_{l=0}^{q-2} G_{-l}G_{2l+\frac{q-1}{2}}G_{-l}T^l\left(\frac{g}{f^2}\right). \tag{28}$$

Replacing $l$ by $l-\frac{q-1}{2}$ we deduce that

$$\begin{aligned} q\cdot(\#E_2(\mathbb{F}_q)-1) &= q^2+\frac{G_{\frac{q-1}{2}}\phi(-1)}{q-1}\sum_{l=0}^{q-2}G_{-l+\frac{q-1}{2}}G_{2l+\frac{q-1}{2}}G_{-l+\frac{q-1}{2}}T^{l-\frac{q-1}{2}}\left(\frac{g}{f^2}\right)\\ &= q^2+\frac{G_{\frac{q-1}{2}}\phi(-g)}{q-1}\sum_{l=0}^{q-2}G_{-l+\frac{q-1}{2}}G_{2l+\frac{q-1}{2}}G_{-l+\frac{q-1}{2}}T^{l}\left(\frac{g}{f^2}\right). \end{aligned} \tag{29}$$

Using Davenport-Hasse relation 2.4 for $m=2$, $\psi=T^{-l}$ and $\psi=T^{2l}$ successively, we have

$$G_{-l+\frac{q-1}{2}}=\frac{G_{\frac{q-1}{2}}G_{-2l}T^{l}(4)}{G_{-l}}$$

and

$$G_{2l+\frac{q-1}{2}}=\frac{G_{4l}G_{\frac{q-1}{2}}T^{-l}(16)}{G_{2l}}.$$

Putting these values in (29) and using lemma 2.1, we obtain

$$q\cdot(\#E_2(\mathbb{F}_q)-1)=q^2+\frac{q^2\phi(-g)}{q-1}\sum_{l=0}^{q-2}\frac{G_{-2l}G_{-2l}G_{4l}}{G_{-l}G_{-l}G_{2l}}T^{l}\left(\frac{g}{f^2}\right).$$

We now put $T=\overline{\omega}$. Then (1) and Gross-Koblitz formula (Theorem 2.5) yield

$$\begin{aligned} a_q(E_2) &= -\frac{q\phi(-g)}{q-1}\sum_{l=0}^{q-2}(-p)^{\sum_{i=0}^{r-1}\{2\langle-\frac{2lp^i}{q-1}\rangle+\langle\frac{4lp^i}{q-1}\rangle-2\langle-\frac{lp^i}{q-1}\rangle-\langle\frac{2lp^i}{q-1}\rangle\}}\\ &\quad\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{4lp^i}{q-1}\rangle\right)}{\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{2lp^i}{q-1}\rangle\right)}\overline{\omega}^l\left(\frac{g}{f^2}\right)\\ &= -\frac{q\phi(-g)}{q-1}\sum_{l=0}^{q-2}(-p)^{s}\overline{\omega}^l\left(\frac{g}{f^2}\right)\\ &\quad\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-2lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{4lp^i}{q-1}\rangle\right)}{\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{-lp^i}{q-1}\rangle\right)\Gamma_p\left(\langle\frac{2lp^i}{q-1}\rangle\right)}, \end{aligned} \tag{30}$$

where $s=\sum_{i=0}^{r-1}\left\{2\langle\frac{-2lp^i}{q-1}\rangle+\langle\frac{4lp^i}{q-1}\rangle-2\langle\frac{-lp^i}{q-1}\rangle-\langle\frac{2lp^i}{q-1}\rangle\right\}$. Next we use lemma 3.1 and after simplification we obtain

$$\begin{aligned}a_q(E_2)=&-\frac{q\phi(-g)}{q-1}\sum_{l=0}^{q-2}(-p)^s\overline{\omega}^l\left(\frac{4g}{f^2}\right)\\&\times\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)}\\&\frac{\Gamma_p\left(\left\langle\left(\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{4}\rangle\right)\Gamma_p\left(\langle\frac{3p^i}{4}\rangle\right)}.\end{aligned}\tag{31}$$

We now simplify the expression for $s$ and find that

$$s=\sum_{i=0}^{r-1}\left\{\lfloor\frac{2lp^i}{q-1}\rfloor+2\lfloor\frac{-lp^i}{q-1}\rfloor-2\lfloor\frac{-2lp^i}{q-1}\rfloor-\lfloor\frac{4lp^i}{q-1}\rfloor\right\}.\tag{32}$$

The following relation for $0\leq l\leq q-2$

$$\begin{aligned}&\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\\&=\prod_{i=0}^{r-1}\Gamma_p\left(\left\langle\left(-\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\end{aligned}$$

and lemma 3.3 yield

$$\begin{aligned}a_q(E_2)=&-\frac{q\phi(-g)}{q-1}\sum_{l=0}^{q-2}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle\frac{p^i}{2}\rangle-\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{p^i}{4}\rangle+\frac{lp^i}{q-1}\rfloor-\lfloor\langle-\frac{3p^i}{4}\rangle+\frac{lp^i}{q-1}\rfloor}\\&\times\frac{\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(\frac{1}{2}-\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)\Gamma_p\left(\langle\frac{p^i}{2}\rangle\right)}\\&\times\frac{\Gamma_p\left(\left\langle\left(-\frac{1}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)\Gamma_p\left(\left\langle\left(-\frac{3}{4}+\frac{l}{q-1}\right)p^i\right\rangle\right)}{\Gamma_p\left(\langle-\frac{p^i}{4}\rangle\right)\Gamma_p\left(\langle-\frac{3p^i}{4}\rangle\right)}\overline{\omega}^l\left(\frac{4g}{f^2}\right)\\&=q\cdot\phi(-g)\cdot{}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{1}{4}, & \frac{3}{4}\end{matrix}\,|\frac{4g}{f^2}\right]_q.\end{aligned}$$

This completes the proof of the theorem. □

**Proof of Theorem 1.4:** Here $j(E_{a,b})\neq 1728$ and hence $b\neq 0$. Let $h\in\mathbb{F}_q^{\times}$ be such that $h^3+ah+b=0$. A change of variables $(x,y)\mapsto(x+h,y)$ takes the elliptic curve $E_{a,b}:y^2=x^3+ax+b$ to

$$E'':y^2=x^3+3hx^2+(3h^2+a)x.\tag{33}$$

Clearly $a_q(E_{a,b}) = a_q(E'')$ and $3h \neq 0$. Using Theorem 3.5 for the elliptic curve $E''$, we complete the proof.

## Acknowledgment

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